NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Mathematics Extension 1
Assessment Task 2
Term 1, 2014

Name:________________________________ Mathematics Class:_______

Student Number _________________________________

Time Allowed: 50 minutes + 2 minutes reading time

Total Marks: 33

Section I – Pages 2 – 3

5 Marks

Attempt Questions 1-5

Answer on Multiple Choice Answer Sheet.

Section II – Pages 5 – 7

28 Marks

Attempt Questions 6 – 8

Show all necessary working.

Question	1,	2,3,4	5	6a	6 b,c	6 d	7-a	7 b	7 c	8-a	8b	Total
PE2		/3		/2								5
HE2						/3		/3				/6
HE6	/1		/1		/5		/3			/5		/15
HE7									/3		/4	/7
												/33

Section I

5 marks
Attempt Questions 1 – 5
Allow about 7 minutes.
Use the multiple choice answer sheet for questions 1-5

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1. When the substitution $u = x - 1$ is used, $\int x\sqrt{x-1}\,dx$ is equivalent to:

   (A) $\int \left(u\sqrt{u} + \sqrt{u}\right)\,du$

   (B) $\int \left(u\sqrt{u} + 1\right)\,du$

   (C) $\int \left(u\sqrt{u} - \sqrt{u}\right)\,du$

   (D) $\int \left(u\sqrt{u} + u\right)\,du$

2. Which of the following is the formula for the *n*th term of a series with the sum $S_n = 3n + 2n^2$?

   (A) $T_n = 6 + 4n$

   (B) $T_n = -4 + 9n$

   (C) $T_n = 5 + 4n$

   (D) $T_n = 1 + 4n$

3. What is the value of $\sum_{n=1}^{\infty} \frac{3^{n-1}}{5^{2n-1}}$?

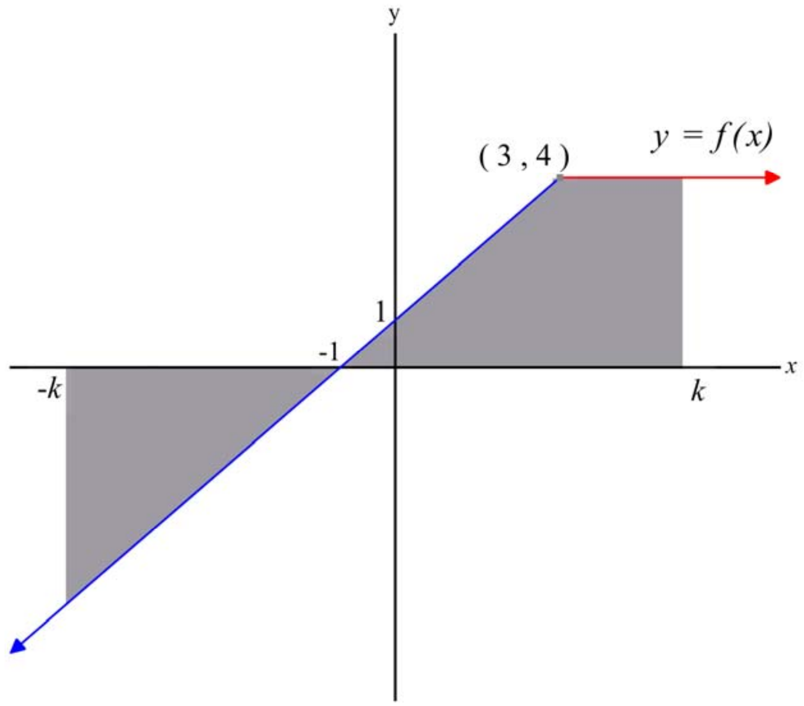
   (A) $\frac{5}{22}$

   (B) $\frac{15}{22}$

   (C) $\frac{25}{122}$

   (D) $\frac{75}{122}$

4. Below is the graph of $y = f(x)$.



What is the value of k so that $\int_{-k}^{k} f(x)\, dx = 0$?

(A) 5

(B) 8

(C) 9

(D) 10

5. A series is defined as $1 + \frac{p^2}{(1-p)^2} + \frac{p^4}{(1-p)^4} + \frac{p^6}{(1-p)^6} + \ldots$ where $p \neq 1$ and $p \neq 0$.

For what values p does the series **NOT** have a limiting sum?

(A) $p \leq -\frac{1}{2}$

(B) $p \geq \frac{1}{2}$

(C) $p \leq -\frac{1}{2}$ and $p \geq \frac{1}{2}$

(D) $-\frac{1}{2} \leq p \leq \frac{1}{2}$

End of Section I

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Section II

28 marks
Attempt Questions 6 - 8
Allow about 43 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra writing paper is available.

In Questions 6-8, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 **(10 Marks)** Use a Separate writing booklet

(a) Show that sum of the series $1+3+5+7+...+(2n-1)$ is a perfect square. **2**

(b) Evaluate $\int_1^4 x\left(2\sqrt{x}-1\right)\,dx$. **2**

(c) Find $\int \frac{x^3}{\left(x^4+1\right)^2}\,dx$ by using the substitution $u=x^4+1$. **3**

(d) Use mathematical induction to prove that: **3**

$$2\times1+3\times2+4\times3+...+n(n-1)=\frac{n\left(n^2-1\right)}{3}$$

for all integers $n\geq2$.

End of Question 6

Question 7 (9 Marks) Use a Separate writing booklet.

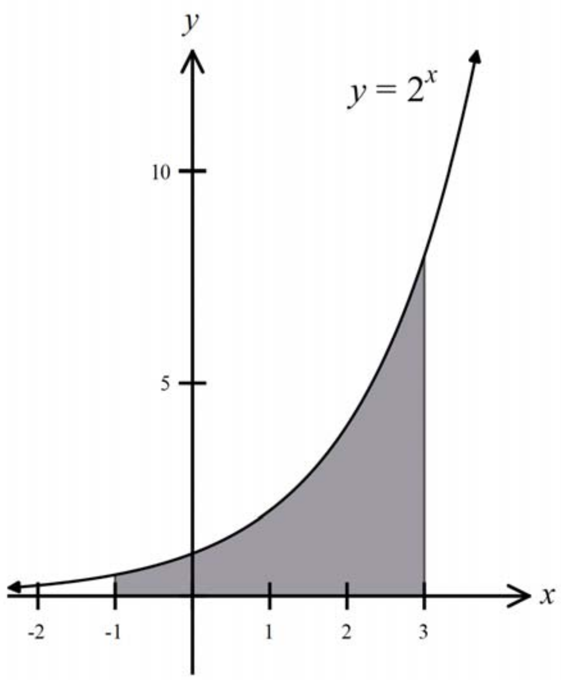
(a) Use the substitution $u^2 = 5 - x$ to evaluate $\int_1^4 (x-1)\sqrt{5-x}\, dx$. **3**

(b) Using the Principle of Mathematical Induction, prove that **3**

$$7^{2n} - 3^{3n} \text{ is divisible by } 11$$

for every integer $n \geq 1$.

(c) In the diagram below, the area under the curve $y = 2^x$ between $x = -1$ and $x = 3$ has been shaded. **3**



Use Simpson's Rule with five function values to determine an approximate volume for the solid of revolution generated when the shaded region is rotated around the *x*-axis.

Round your answer off to four significant figures.

End of Question 7

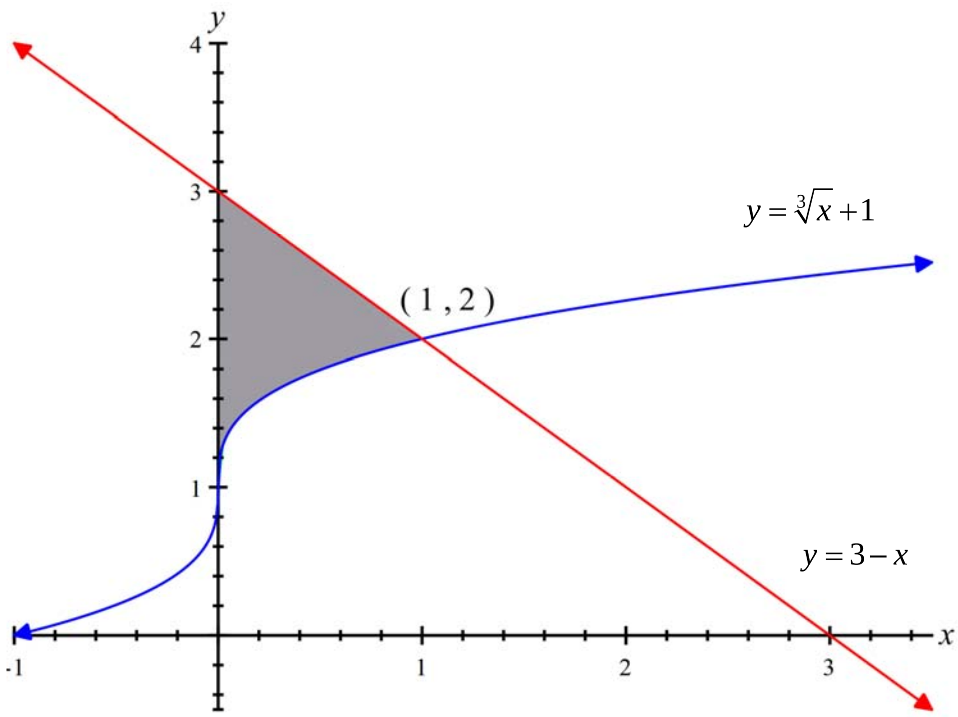
Question 8 **(9 Marks)** Use a Separate writing booklet

(a) (i) Differentiate $(x-2)(x+4)^7$. Fully factorise your answer. **2**

(ii) Hence find $\int (4x-5)(x+4)^6 dx$. **1**

(iii) Hence or otherwise find $\int 4x(x+4)^6 \; dx$. **2**

(b) In the diagram below, the line $y = 3 - x$ meets the graph of $y = \sqrt[3]{x} + 1$ at $(1, 2)$.



The region between $y = 3 - x$, $y = \sqrt[3]{x} + 1$ and the y-axis is shaded.
This shaded region is then rotated around the y-axis to form a solid of revolution.

Find the volume of the solid generated. **4**

End of Examination

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HSC Examination – Mathematics Extension 1 Task 2 2014

Multiple Choice Answer Sheet

Name ________________________________

Completely fill the response oval representing the most correct answer.

1.	A ⬭	B ⬭	C ⬭	D ⬭
2.	A ⬭	B ⬭	C ⬭	D ⬭
3.	A ⬭	B ⬭	C ⬭	D ⬭
4.	A ⬭	B ⬭	C ⬭	D ⬭
5.	A ⬭	B ⬭	C ⬭	D ⬭

SECTION I – Multiple Choice

1. $\int x\sqrt{x-1}\ dx$ is equivalent to:

$u = x-1$

$x = u+1$

$\frac{du}{dx} = 1$

$du = dx$

$\int (u+1)\sqrt{u}\ du = \int \left(u\sqrt{u} + \sqrt{u}\right)\ du$

Answer is A

2. If $S_n = 3n + 2n^2$, what is T_n

METHOD 1

$T_n = S_n - S_{n-1}$

$= \left(3(n) + 2(n)^2\right) - \left(3(n-1) + 2(n-1)^2\right)$

$= 3n + 2n^2 - 3n + 3 - 2n^2 + 4n - 2$

$= 1 + 4n$

METHOD 2

$S_1 = 3(1) + 2(1)^2$

$= 5$

$T_1 = 5$

and

$S_2 = 3(2) + 2(2)^2$

$= 14$

$T_2 = 14 - 5$

$= 9$

$\therefore a = 5$ and $d = 4$

$T_n = 5 + (n-1)4$

$= 1 + 4n$

Answer is D

3. What is the value of $\sum_{n=1}^{\infty} \frac{3^{n-1}}{5^{2n-1}}$?

When $n = 1$

$\frac{3^0}{5^1} = \frac{1}{5}$

When $n = 2$

$\frac{3^1}{5^3} = \frac{3}{125}$

When $n = 3$

$\frac{3^2}{5^5}$

$\therefore \sum_{n=1}^{\infty} \frac{3^{n-1}}{5^{2n-1}}$ is a sum of a geometric series

with first term $\frac{1}{5}$ and common ratio $r = \frac{3}{5^2}$

$\therefore S_\infty = \frac{a}{1-r} = \frac{\frac{1}{5}}{1-\frac{3}{5^2}} = \frac{\frac{1}{5}}{\frac{22}{25}} = \frac{5}{22}$

Answer is A

4. Find k if $\int_{-k}^{k} f(x)\ dx = 0$?

$AREA_{TRAPEZIUM} = AREA_{TRIANGLE}$

$\frac{(k+1)+(k-3)}{2} \times 4 = \frac{(k-1)^2}{2}$

METHOD 1

Test the possible values for k

METHOD 2

Solve the equation for k

$(4k-4) = \frac{k^2 - 2k + 1}{2}$

$8k - 8 = k^2 - 2k + 1$

$0 = k^2 - 10k + 9$

$0 = (k-9)(k-1)$

$k = 9$ *or* 1

But $k > 3$

$k = 9$

Answer is C

5. Not a limiting sum if common ratio r is

$r \le -1$ or $r \ge 1$

$$r = \frac{p^2}{(1-p)^2}$$

$and\ \frac{p^2}{(1-p)^2}$ is alway positive or zero.

So solve

$$\frac{p^2}{(1-p)^2} \ge 1$$

$$p^2 \ge (1-p)^2$$

$$p^2 \ge 1-2p+p^2$$

$$0 \ge 1-2p$$

$$2p \ge 1$$

$$p \ge \frac{1}{2}$$

Answer is B

QUESTION 6

(a)

$$1+3+5+7+...+(2n-1)$$

$$a=1 \quad d=2$$

$$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$$

$$S_n = \frac{n}{2}\left[2(1)+(n-1)2\right]$$

$$S_n = \frac{n}{2}[2n]$$

$$S_n = n^2$$

(b) Evaluate $\int_1^4 x\left(2\sqrt{x}-1\right)\,dx$.

$$\int_1^4 x\left(2\sqrt{x}-1\right)\,dx$$

$$\int_1^4 \left(2x\sqrt{x}-x\right)\,dx$$

$$\int_1^4 \left(2x^{\frac{3}{2}}-x\right)\,dx$$

$$\left[\frac{4x^{\frac{5}{2}}}{5}-\frac{x^2}{2}\right]_1^4$$

$$\left(\frac{4(4)^{\frac{5}{2}}}{5}-\frac{(4)^2}{2}\right)-\left(\frac{4(1)^{\frac{5}{2}}}{5}-\frac{(1)^2}{2}\right)$$

$$\left(\frac{128}{5}-8\right)-\frac{8-5}{10}$$

$$\frac{173}{10}$$

(c) Find $\int \frac{x^3}{\left(x^4+1\right)^2}\,dx$ by using $u = x^4+1$.

$$\int \frac{x^3}{\left(x^4+1\right)^2}\,dx$$

$$\int \frac{1}{4(u)^2}\,du$$

$$u = x^4+1$$

$$\frac{du}{dx} = 4x^3$$

$$dx = \frac{du}{4x^3}$$

$$\int \frac{1}{4(u)^2}\,du = \int \frac{u^{-2}}{4}$$

$$= -\frac{u^{-1}}{4}+C$$

$$= -\frac{1}{4u}+C$$

$$= -\frac{1}{4\left(x^4+1\right)}+C$$

(d)

$$LHS = 2\times 1 = 2$$

$$RHS = \frac{2\left(2^2-1\right)}{3} = 2$$

$\therefore$ True for $n=2$

Assume true for $n=k$

$$2\times 1+3\times 2+4\times 3+...+k(k-1) = \frac{k\left(k^2-1\right)}{3}$$

Prove that if true for $n=k$

then true for $n=k+1$

AIM : prove that:

$$2\times 1+3\times 2+4\times 3+...+k(k-1)+(k+1)k$$

$$=\frac{(k+1)\left((k+1)^2-1\right)}{3}$$

$$=\frac{(k+1)(k^2+2k)}{3}$$

$$=\frac{k(k+1)(k+2)}{3}$$

$$LHS=2\times 1+3\times 2+4\times 3+...+k(k-1)+(k+1)k$$

$$=\frac{k(k^2-1)}{3}+(k+1)k$$

$$==\frac{k(k-1)(k+1)}{3}+(k+1)k$$

$$=\frac{(k+1)k}{3}[k-1+3]$$

$$==\frac{(k+1)k}{3}[k+2]=RHS$$

Question 7 (9 Marks)

(a) Use $u^2=5-x$ to evaluate $\int_1^4 (x-1)\sqrt{5-x}\ dx$

$$u^2=5-x$$

$$x=5-u^2$$

$$\frac{dx}{du}=-2u$$

$$dx=(-2u)du$$

$$x=4 \quad u=\sqrt{5-4}=1$$

$$x=1 \quad u=\sqrt{5-1}=2$$

$$\int_1^4 (x-1)\sqrt{5-x}\ dx$$

$$\int_2^1 (5-u^2-1)\sqrt{u^2}\ (-2u)du$$

$$-2\int_2^1 (4-u^2)(u)(u)du$$

$$-2\int_2^1 (4u^2-u^4)du$$

$$-2\left[\frac{4u^3}{3}-\frac{u^5}{5}\right]_2^1$$

$$-2\left[\left(\frac{4(1)^3}{3}-\frac{(1)^5}{5}\right)-\left(\frac{4(2)^3}{3}-\frac{(2)^5}{5}\right)\right]$$

$$=6\frac{4}{15}$$

(b) Show that the result is true for $n=2$

$$7^{2(1)}-3^{3(1)}=49-27$$

$$=22$$

Which is divisible by 11

$\therefore$ True for $n=1$

Assume true for $n=k$

$7^{2k}-3^{3k}=11M$ Where M is an integer.

$7^{2k}=11M+3^{3k}$

Prove that if true for $n=k$

then true for $n=k+1$

AIM : prove that:

$7^{2(k+1)}-3^{3(k+1)}=11N$ Where N is an integer.

$$LHS=7^{2(k+1)}-3^{3(k+1)}$$

$$=7^{2k+2}-3^{3k+3}$$

$$=49\times 7^{2k}-27\times 3^{3k}$$

$$=49\times\left(11M+3^{3k}\right)-27\times 3^{3k} \quad \text{From assumption}$$

$$=49\times 11M+49\times 3^{3k}-27\times 3^{3k}$$

$$=49\times 11M+22\times 3^{3k}$$

$$=11\left(49M+2\times 3^{3k}\right)$$

$$=11N \qquad \because \left(49M+2\times 3^{3k}\right) \text{ is an integer}$$

(c) Use Simpson's Rule with five function values to determine an approximate volume.

x	-1	0	1	2	3
2^x	$\frac{1}{2}$	**1**	**2**	**4**	**8**
2^{2x}	$\frac{1}{4}$	**1**	**4**	**16**	**64**

k	**1**	**4**	**2**	**4**	**1**

$k2^{2x}$	$\frac{1}{4}$	**4**	**8**	**64**	**64**

$$\sum ky^2 = 140\frac{1}{4}$$

$$Volume = \pi\int_{-1}^{3} y^2 dx$$
$$= \pi\frac{1}{3}\left(140\frac{1}{4}\right)$$
$$= \pi \times 46\frac{3}{4}$$
$$= \frac{187\pi}{4} = 146.9 units^3$$

Question 8

(a) $$\frac{d}{dx}(x-2)(x+4)^7 = (x+4)^7 + 7(x-2)(x+4)^6$$
$$= (x+4)^6\left[x+4+7x-14\right]$$
$$= (x+4)^6(8x-10) \quad .$$

(b) (ii) Hence find $\int (4x-5)(x+4)^6 dx$.

$$\int (x+4)^6(8x-10) = (x-2)(x+4)^7 \quad \text{From part } (a)$$
$$\int (x+4)^6(4x-5) = \frac{1}{2}(x-2)(x+4)^7$$

(iii) Hence find $\int 4x(x+4)^6 \, dx$.

$$\int (4x-5)(x+4)^6 = \frac{1}{2}(x-2)(x+4)^7 + C \qquad \text{From part } (b)$$
$$\int 4x(x+4)^6 - \int 5(x+4)^6 = \frac{1}{2}(x-2)(x+4)^7 + C$$
$$\int 4x(x+4)^6 = \frac{1}{2}(x-2)(x+4)^7 + \int 5(x+4)^6 + C$$
$$= \frac{1}{2}(x-2)(x+4)^7 + \frac{5(x+4)^7}{7} + D$$

(b) Find the volume of the solid generated.

$y = 3 - x$

$x = 3 - y$

$y = \sqrt[3]{x} + 1$

$y - 1 = \sqrt[3]{x}$

$(y-1)^3 = x$

$$V=\pi\int_2^3(3-y)^2\ dy+\pi\int_1^2(y-1)^6\ dy$$

$$=\pi\left[-\frac{(3-y)^3}{3}\right]_2^3+\pi\left[\frac{(y-1)^7}{7}\right]_1^2$$

$$=\pi\left[\left(-\frac{(3-(3))^3}{3}\right)-\left(-\frac{(3-(2))^3}{3}\right)\right]+\pi\left[\left(\frac{((2)-1)^7}{7}\right)-\left(\frac{((1)-1)^7}{7}\right)\right]$$

$$=\pi\left[(0)+\left(\frac{1}{3}\right)\right]+\pi\left[\left(\frac{1}{7}\right)-(0)\right]$$

$$=\frac{10\pi}{21}Units^3$$